Some calculus affordances of a graphics calculator

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Introduction

Calculus at the secondary school level has traditionally represented the peak of school mathematics in Australia, and has been available only to the most capable students. Until recently, many calculus curricula have focused on developing standard techniques, such as those concerned with differentiation and integration, with an emphasis on symbolic procedures for carrying these out in a range of situations. These characteristics have sometimes been inadvertently reinforced by external assessment agencies.

This paper is concerned with exploring the possible ways in which one kind of technology, the graphics calculator, might be productively used by calculus students and their teachers. The paper provides an analysis to describe some affordances provided by graphics calculators: opportunities for significant changes to the teaching and learning of calculus. It is recognised that access to such affordances is not by itself sufficient to bring about changes in practice. However, the paper provides an outline of the main possibilities offered by the calculator, with some examples to illustrate these. One purpose for doing this is to counter the claim, still sometimes heard, that use of calculators is concerned only with students pushing buttons rather than learning to understand mathematics.

Pierce and Stacey (2008) described some affordances of CAS for mathematics education. The present paper highlights that the less sophisticated technology of the graphics calculator (without the symbolic manipulation elements of CAS) provides many opportunities for both students and teachers for the particular mathematical domain of the calculus. The extra affordances provided by CAS calculators are not dealt with here. Rather, the intention is to demonstrate that there is much to gain from exploiting the graphical and numerical capabilities of a modern graphics calculator.

The graphics calculator

The choice of a graphics calculator for learning calculus is motivated by a view that this represents the best prospect for providing widespread access to technology for mathematics within typical schools. This view has been elaborated elsewhere (e.g., by Bradley, Kemp & Kissane (1994)) and so will not be repeated in detail here. The main parts of the argument are that graphics calculators are relatively inexpensive, very portable and include significant educationally valuable software. In contrast, other forms of technology have significant barriers to widespread access. Computers are significantly more expensive than calculators, often depend on easy access to computer laboratories, often require relatively expensive computer software, are not yet available widely enough to be taken into account by curriculum developers and are very unlikely to be permitted for use in important examinations. The Internet, promising in many ways, suffers most of the limitations of computers, and has the additional problem of securing adequate Internet access when and where it is needed.

This is not to say that the graphics calculator is an optimum choice; in the absence of the constraints of the real educational world, computers have the potential to provide a more powerful environment, enriched by large, fast and colourful screens and supported by powerful and multi-purpose software. However, the graphics calculator continues to represent the best compromise between the latest technology and what is likely to be manageable when resources are limited.

A consequence of these arguments is that the graphics calculator in 2008 continues to be the most likely way for senior secondary school students and their teachers to integrate the use of technology into the teaching and learning program for all. Mathematics curricula in most Australian states have now recognised the significance of technology for student learning, typically through the process of permitting some level of graphics calculator use in high-stakes external examinations. Indeed, it is now more than a decade since the US College Board approved the use of graphics calculators on the Advanced Placement Calculus examinations, also recognising the educational advantages of doing so (Kennedy, 2002). This method of encouraging calculator use has at times encouraged a view that the calculator's role is centred on examination use — which is not the case. This paper offers insights into the relevance of the graphics calculator for learning calculus, irrespective of examinations.

Concept representation on graphics calculators

Although there is an emphasis on procedures in calculus in some classrooms, and an unavoidable emphasis on formal procedures in examinations, the concepts of calculus are the most important elements, and are the focus of this paper. Procedural competence can best be developed when students understand the underlying ideas well. In this section of the paper, we consider

how some key concepts of the calculus are represented by graphics calculators, in order to illustrate how educational experiences might be designed differently by teachers than in situations for which technology is not available. Space precludes a complete treatment of possibilities, many of which are described in some detail in Kissane and Kemp (2006). A recent graphics calculator, the Casio fx-9860G, is used for all of the examples below.

Differentiation from first principles

Before the availability of technology, a standard approach to the idea of a derivative involved the well-known first principles definition. While this approach made sense for some students, for many others, the symbolic representations involved were difficult to understand, and the idea of a limit exaggerated the problem. On a graphics calculator, the idea can also be represented, but with an emphasis on the graphical representations and their interpretation. Figure 1 shows one way of doing this for the case of $f(x) = \sin x$, by graphing the difference quotient $[\sin(x+h) - \sin x)]/h$ for a 'small' value of h, in this case $h = 0.1$.

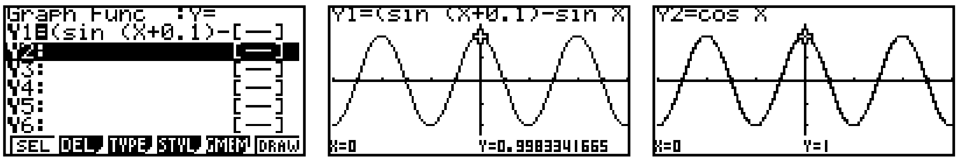

Figure 1. Approximate differentiation of f(x) = sin x by first principles.

In this case, the graph of the quotient function should look familiar to students. The final screen shows that it is very close to the graph of $f(x) = \cos x$, despite the value of $h = 0.1$ not being very 'small.' Students can explore in more detail what happens with smaller values of h, in order to get an intuitive feel for the limiting process, but without the symbolic manipulation requirements — which can be approached at a later stage.

Local linearity

With a graphics calculator available, the first principles definition may not be the best place to start building the concept of differentiation. A key concept underpinning the idea of the derivative of a function at a point is that of local linearity: that, on a small enough interval, the graph of a differentiable function can be well approximated by a line. The approach relies on students having an understanding of the slope of a line, which is always handled before students begin studying the calculus, and then building on this idea to consider the idea of the slope of a curve. Maschietto (2004) described productive work in classrooms using this powerful idea, accessed through a process of 'zooming in' on graphs on calculators. Figure 2 shows the beginning of an example, which can be readily continued by repeatedly zooming in to produce increasingly good approximations to a line around $x = -1$, although it is clear at first that the graph is parabolic and hence is curved.

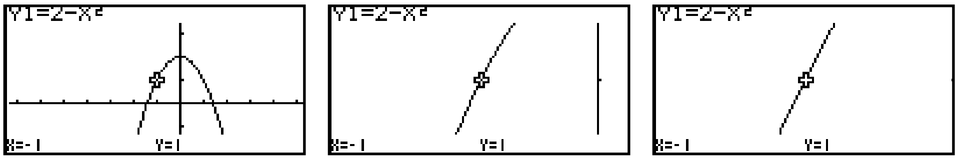

Figure 2. Successive approximations to local linearity of f(x) = 2 – x2 at x = –1 by zooming in.

As well as helping with the idea of linearity, activity of this kind affords access to the key idea of the rate of change of a function at a point (as distinct from the rate of change of a tangent to a function at a point), as suggested in the next section.

Numerical derivative at a point

The concept of a derivative at a point involves the notion of a limit, which is notoriously difficult for beginning students (and was not in fact developed formally in the early years of the calculus by Newton and Leibniz, but appeared instead many years later). A conceptually easier idea involves a rate of change over a small interval, provided by the numerical derivative command of graphics calculators. This concept is closely related to that of local linearity, as it represents the slope of the (local) line. Figure 3 shows an example of the calculator's affordance here: as students trace the graph of the function, both the coordinates of each point and the numerical derivative at the point are provided. A significant conceptual advantage of this image is that it allows for the idea of the rate of change of the function itself, rather than that of tangents to the function, described briefly in the next section.

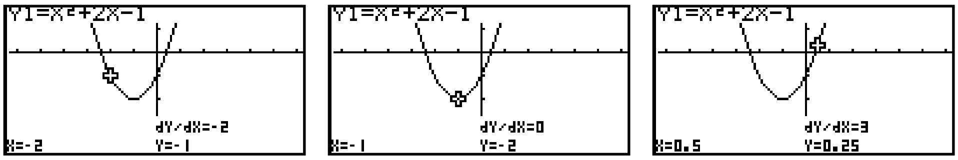

Figure 3. Numerical derivatives of $f(x) = x^2 + 2x - 1$ obtained by tracing.

Similarly, numerical derivatives can be obtained in a table of numerical values of a function. In the example shown in Figure 4, it is clear that the derivatives are steadily increasing as *x* increases, and the idea of the derivative itself being a function is suggested by the changes in values in the table.

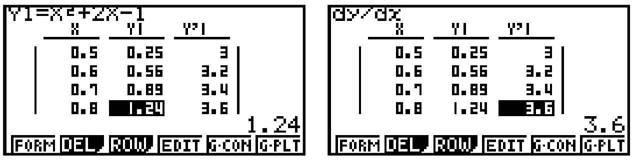

Figure 4. Numerical derivatives of $f(x) = x^2 + 2x - 1$ obtained in a table of values.

Using these capabilities, students can explore many aspects of the relationships between functions, graphs and derivatives, laying important groundwork for a more formal symbolic study of these at a later stage.

Tangents

Traditional approaches to the idea of a derivative at a point have involved a secant that eventually becomes a tangent in the limiting case of the two end-points of the secant coinciding. On a graphics calculator, tangents can be drawn at a point, allowing for a version of this concept image to be experienced by students, and also allowing for a line with a particular slope to be seen (rather than just the numerical value of the slope). Figure 5 illustrates this, using the same function and points as for Figure 3.

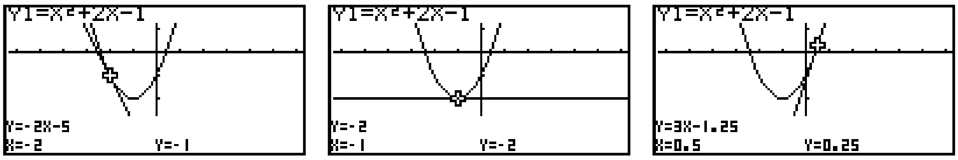

Figure 5. Tangents to $f(x) = x^2 + 2x - 1$ obtained by tracing.

In this case, the screen shows both the tangents at the chosen points as well as their equations. Images like these seem likely to help students connect the rate of change of the function at a point with the tangents to the curves at the point.

Derivative functions

As suggested above, the concept of a derivative function is a natural extension of the idea of a derivative of a function at a point: a derivative function describes the entire family of derivatives at any point for a particular function. On a graphics calculator, the standard symbol for a derivative is used (d/dx), and both a graphical and a numerical representation of the derivative function can be obtained for any function, using a “function transformer” (Kissane & Kemp, 2006, p. 39). In the example shown in Figure 6, the second function $Y2(x)$ is defined as a transformation of the first function, $Y1(x)$. In this case, the transformation involves finding the (numerical) derivative, so that, whenever $Y1(x)$ is changed, the derivative function in $Y2(x)$ is changed automatically.

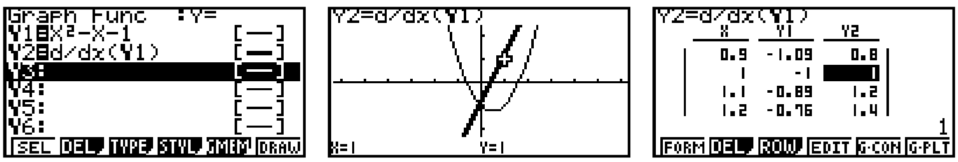

Figure 6. Using a function transformer to represent a derivative function.

These representations, linking functions and their derivatives through simultaneous displays, afford many opportunities for classroom exploration. For example, student attention can be drawn to the significant features of a graph (e.g., slope, turning points) and the associated values of the derivative function. There are possibilities for individual, small group and whole class work (the latter using a public display screen), with good activities available through making changes to the function $Y1(x)$. For example, the case shown in Figure 6 suggests that the derivative function is linear for the chosen quadratic function, supported both graphically and numerically. Student exploration will reveal that this is the case for all the quadratic functions they enter, not only this one, and may lead to the conjecture that this is the case for all quadratic functions, not just those tested.

The examples shown in Figure 7 suggest some ways in which activity of this kind might be productively used. For each screen the scales are the same: the tick marks on the axes are at every unit, so that all three screens have the same domain and range. Each screen has a different function defined as $Y1(x)$, with the derivative transformer applied in each. The first screen shows $Y1(x) = x^2 + 2x - 2$ and its derivative transformation, while the second screen shows $Y1(x) = x^2 + 2x - 3$ and its derivative transformation. These two screens show that two different functions can have the same derivative function, which is an important idea fundamental to the later study of differential equations or of anti-differentiation generally.

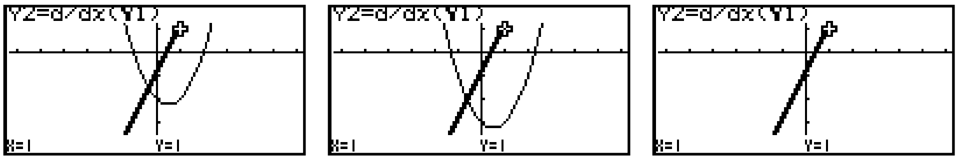

Figure 7. Different functions can have the same derivative function.

The final screen in Figure 7 has been obtained by turning off the graphing for function $Y1(x)$ and graphing only its derivative function $Y2(x)$; this suggests an activity in which students are asked to imagine (or sketch, or describe) the function that might have a derivative as shown.

Optimisation

Historically, a major motivation for differential calculus involves optimising situations, through the location of relative maxima and minima of functions. Students can now encounter these ideas earlier than using analytical calculus, through the agency of graphics calculators, so that the formal study of the calculus might be seen nowadays as an opportunity to formalise the processes involved, including seeking general or exact solutions to optimisation problems. Indeed, in some (extreme) cases, unlikely to be encountered by beginners, the technology may be insufficient for the task; Dubinsky (1995) showed a good example of this, reminding us of the necessity of analysis for some tasks.

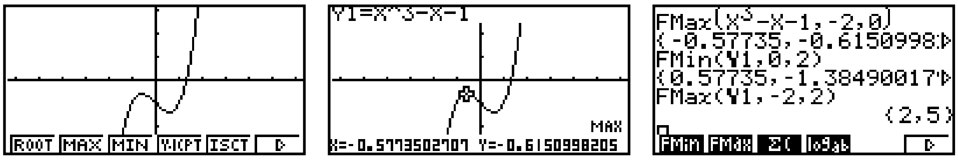

Figure 8. Obtaining relative maxima and minima from a graph or numerically.

At first sight, motivation might seem to be undermined when students have access to a graphics calculator which allows for very good approximations to maximum and minimum points to be obtained visually, or numerically, using maximum and minimum commands, as suggested by Figure 8.

Even though the values of the relative maxima and minima can be obtained through calculator commands, the graphics calculator offers ways of linking concepts together, rather than merely producing numerical answers to optimisation questions. To illustrate this, the first screen in Figure 9 shows the same function as Figure 8, but with the value of the derivative showing at the same time. This sort of activity seems likely to help students see the connections between the value of the derivative and the turning point. The latter two screens show a check on the exact turning point at $x = 1/\sqrt{3}$.

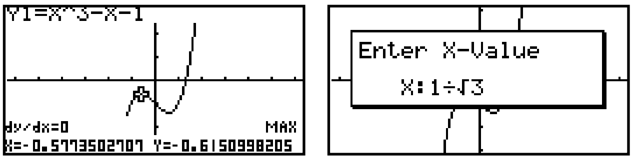

Figure 9. Calculator explorations associated with turning points.

In addition, other connections are made possible by the graphics calculator. Figure 10 shows another way of connecting the turning points with the derivative, by locating a root of the (numerical) derivative function in order to find a local maximum and using the calculator trace to check that there seems to be a relative minimum at $x = 1/\sqrt{3}$.

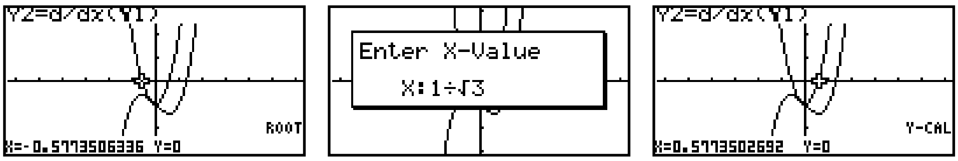

Figure 10. Further calculator explorations related to optimisation.

The examples provided here for this function suggest several ways in which the experiences of both students and teachers may be altered, hopefully supported and enriched, through using graphics calculator features.

Discontinuity

Introductory students of calculus generally encounter continuous functions, although some elementary functions have points of discontinuity. By the (digital) nature of the technology, continuity is only approximated in graphical representations, both on calculators and on computers, for which screens are comprised of many discrete pixels. Despite this limitation, concepts of continuity and discontinuity can be developed with the assistance of technology. The rational function shown in Figure 11 is a good example.

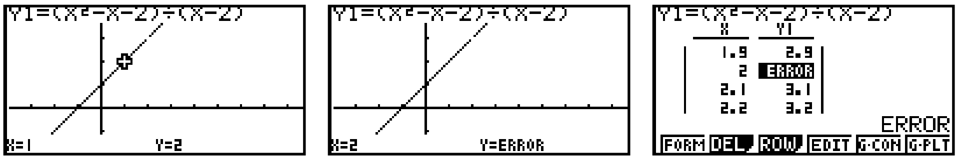

Figure 11. Representing a rational function that is discontinuous at $x = 2$.

The ‘hole’ in the line on the screen suggests the idea of a removable discontinuity and the error shown in the table of values reinforces this. Successive ‘zooming in’ allows students to experience the nature of the discontinuity, which is evident only at the exact value of $x = 2$, and not at points either side of this. Figure 12 shows an example of this.

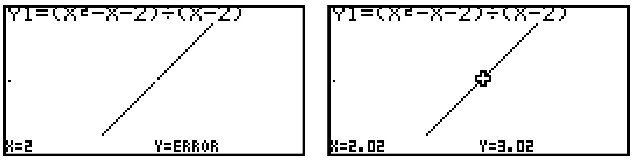

Figure 12. Zooming in graphically on a point of discontinuity.

Again, this sort of affordance can help build the concept of discontinuity, both for students exploring examples on their own calculators or a class exploring suitable examples with the support of a projected calculator. Similarly, Figure 13 shows the same idea numerically, using a table of values, rather than a graph, to represent the function.

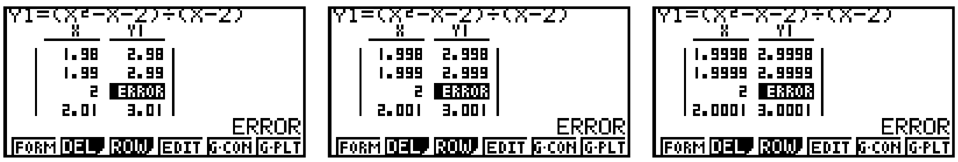

Figure 13. Zooming in numerically on a point of discontinuity.

Both the graphical and numerical representations afford students the opportunity to explore the idea of getting ‘closer and closer’ to an important point — but not quite getting there, an important idea that underpins the conceptually difficult idea of a limit.

Limits and asymptotic behaviour

As noted earlier, notions of limiting behaviour, while critical to understanding the calculus, can be dealt with informally at the early stages. Intuitions regarding the idea of a limit can be developed on a calculator by choosing smaller and smaller values of a variable (in the case of a limit as $x \to 0$) or larger and larger values of a variable (in the case of a limit as $x \to \infty$). Figure 14 shows the example of the limit, as $x \to 0$ of $(\sin x)/x$.

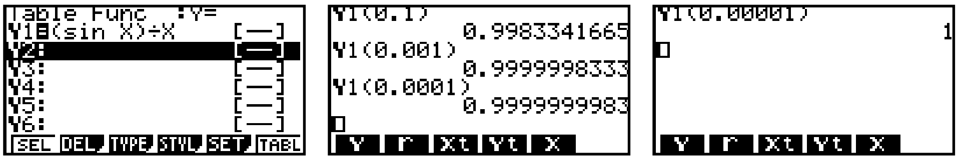

Figure 14. Numerical approximations to a limit as $x \to 0$ of $(\sin x)/x$.

Once the appropriate function has been defined (in the graph or table area of the calculator), the function can be evaluated at will for any values. In the case of $(\sin x)/x$, as the value of x approaches zero, the function appears to get 'closer and closer' to 1. As the final screen shows, the graphics calculator has a limitation that it will eventually round the result to give the impression that the value is 1 (which is not the case). Rather than regard this as an 'error', it affords a good opportunity for students to understand the differences between the (continuous, infinite) mathematical ideas involved and their representation on a (discrete, finite) machine.

Similarly, Figure 15, shows examples of the limit as $x \to \infty$ of $(1 + 1/x)^x$, which is e. Once again, the final screen shows that the calculator eventually gives the impression that the value is exactly equal to e, when x is large enough. Recognising these as problems of representation allows students to begin to appreciate the subtle differences between mathematical ideas and their representation on a finite, discrete machine.

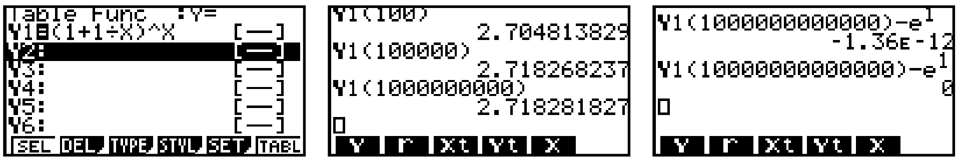

Figure 15. Numerical approximations to a limit as $x \to \infty$ of $(1 + 1/x)^x$.

Graphical approximations of both limiting and asymptotic behaviour are also available and likely to help develop student insight, but space precludes discussing these further here.

Convergence of a series

As for limits, convergence is a problematic concept, as it also involves notions of the infinite. A graphics calculator can offer some conceptual support, both graphically and numerically, by allowing an easy mechanism to generate and add many terms quickly. The example shown in Figure 16 converges very

rapidly to *e* (much more so than that shown in Figure 14) and is easily constructed and explored by students familiar with the graphics calculator.

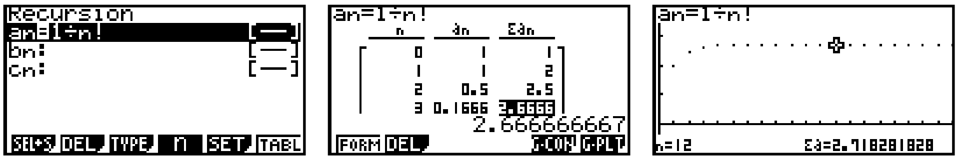

Figure 16. Exploring the convergence of the exponential series numerically and graphically.

Graphs of asymptotes

Asymptotic behaviour of functions is well-represented on graphics calculators, affording students opportunities to explore the ideas directly. Figure 17 shows an example with graphs of the rational function $f(x) = (x - 1)/(x - 2)$. The asymptotic behaviour can be explored by 'zooming out' as in the middle screen or by manually changing the x-axis bounds, as in the third screen (for which the boundaries are –200 and 200). The long term behaviour is clear visually, with $f(x)$ approaching 1 in both directions. This kind of experience is complemented by algebraic representation of the function as $f(x) = 1 + 1/(x - 2)$, making it clear that $f(x) \to 1$ as $x \to \infty$.

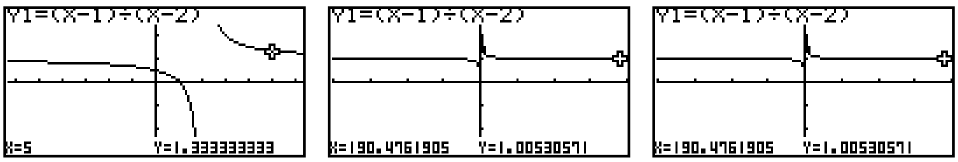

Figure 17. Exploring the asymptotic behaviour of f(x) = (x – 1)/(x – 2).

Similarly, Figure 18 shows an example of a function $f(x) = x + 1/(x - 2)$ with an oblique asymptote. The first screen shows this visually for x-values between –200 and 200, making it clear that the function seems to approach the identity function $f(x) = x$ asymptotically.

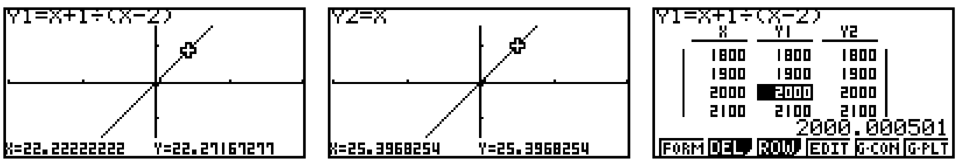

Figure 18. Exploring the asymptotic behaviour of f(x) = x + 1/(x – 2).

The second screen shows the graph of $f(x)$ together with that of the identity function; they are virtually indistinguishable at this scale. The final screen shows how the ideas can be explored numerically, making it clear that, although the graphs cannot be distinguished graphically, the values are not quite the same. Informal explorations of these kinds afford new opportunities for students to appreciate the concepts involved.

Differential equations

A powerful way of thinking about elementary differential equations uses a slope-field diagram, such as that shown in Figure 19. While some more recent technologies (such as Casio's Classpad 330) routinely provide a mechanism for drawing such a diagram, in this case two small calculator programs are needed. (Kissane & Kemp, 2006, pp. 277–279) Together with earlier experience (such as that shown in Figure 7 above) showing that many different functions can have the same derivative function, calculator explorations with examples like this will help students to appreciate that many possible functions can match the direction field displayed, depending on the initial conditions.

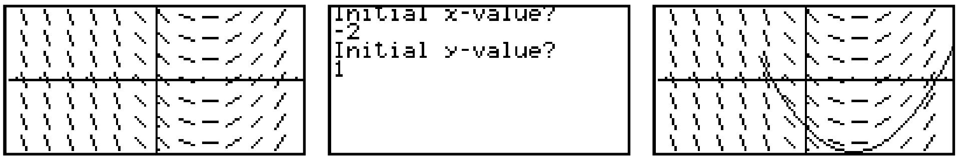

Figure 19. Representing and solving the differential equation dy/dx = x/2 – 1, with boundary condition (–2,1).

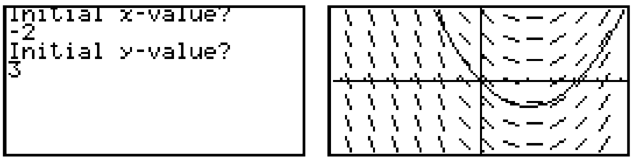

Figure 20. Representing and solving the differential equation dy/dx = x/2 – 1, with boundary condition (–2,3).

In this case, the slope-field diagram provides a good mental picture of the idea of an elementary differential equation, and the idea that different boundary conditions lead to different solutions. Importantly, the various solutions are part of a larger family of solutions, in this case, a family of quadratic functions.

Appropriate use of technology

The graphics calculator provides new affordances for teaching and learning calculus, but this is no guarantee that they will be used well by either students or teachers. When graphics calculators first became available, there were already significant discussions regarding the best ways to teach calculus and what to teach, especially in the USA, where discussions of these kinds were referred to under the general heading of 'calculus reform.' For example, Cipra (1988, p. 1492) reported a concern that, despite promises that more time might be spent on the underlying ideas, in fact time was being spent on showing students which button to push. Similarly, Klein and Rosen (1997), responding to Mumford (1997) suggested that insufficient rigour and over-

use of calculators were associated with the calculus reform movement. Such debate indicates that care needs to be taken to incorporate technology, including graphics calculators in particular, intelligently into the curriculum.

Similarly, in highlighting the ways in which teachers make use of technology in classrooms, Goos (2006) has made clear the many effects of contexts (including those in a school, particular to a person, related to an official curriculum or examination, and so on). While the graphics calculator offers new opportunities, teachers will need both support and help to make good use of these to support student learning. Part of the support will include textbooks, curricula and examination systems that recognise the possibilities and allow them to be used.

Both Solow (1994) and Stick (1997) have reported favourable results from using graphics calculators in college classrooms in the USA, while Rochowicz (1997) reported a survey of college faculty that was mostly positive about such practices. It is clear from the field, from these and other researchers, that graphics calculators are no panacea, and require significant effort by teachers to use them well. The work of researchers like Guin and Trouche (1999) provides important elaborations of the process of 'instrumental genesis,' concerned with rendering a calculator an efficient and effective tool for developing student understanding.

Conclusion

The graphics calculator offers both teachers and students a range of opportunities to encounter the calculus in a fresh way. In this paper, the major kinds of affordances associated with the calculator are identified and illustrated, and it is suggested that energy spent on taking advantage of these in classrooms will be worthwhile. The focus of this work should be on the development of the important concepts of the calculus.

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